Unit: 6

Line Integral

Line Integral: Let $\vec{F}$ is a vector function then integral $I = \int \vec{F} \cdot d\vec{r}$ is called as line integral of $\vec{F}$ over some curve C.

Ques: Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$

(1) $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, C is line segment from $(1,2,2)$ to $(3,6,6)$

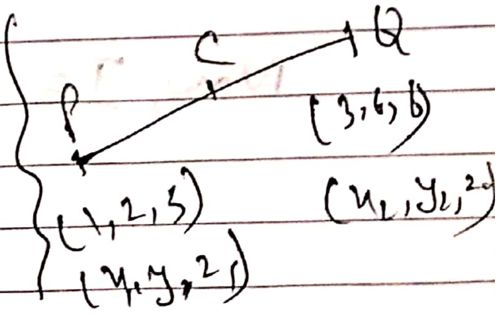
Sol^n: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz.$$

Now, integral $\vec{F} \cdot d\vec{r} = x\,dx + y\,dy + z\,dz$

$$I = \int_C \vec{F} \cdot d\vec{r} = \int_C (x\,dx + y\,dy + z\,dy)$$

eq. of line C is



$$\frac{x-1}{3-1} = \frac{y-2}{6-2} = \frac{z-2}{6-2}.$$

$$= \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} = \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-2}{2}$$

Now $\frac{x-1}{1} = \frac{y-2}{2} \Rightarrow y-2 = 2x-2$ $\quad dy = 2dx$

$\boxed{y = 2x}$

and $\frac{x-1}{1} = \frac{z-2}{2} \Rightarrow z-2 = 2x-2 \Rightarrow \boxed{z = 2x}$

$z = 2dx.$

$$I = \int_{x=1}^{3} \left[x\,dx + 2x(2dx) + 2x(2dx) \right]$$

$$= \int_1^3 9x\,dx = \left. \frac{9x^2}{2} \right|_1^3$$

$$= \frac{9\times 9}{2} - \frac{9}{2} = \frac{81}{2} - \frac{9}{2} = \frac{72}{2} = 36.$$

or

or From eq. of line, Consider $= \frac{x-1}{1} = \frac{y-2}{2}$

$2x-2 = y$ $\quad \boxed{x = \frac{y}{2}}$ $\quad dx = \frac{dy}{2}.$

again, $\frac{y-2}{2} = \frac{z-2}{2} \Rightarrow z = y.$ $\quad \boxed{dz = dy}$

Now, $I = \int_c (x\,dx + y\,dy + z\,dz)$

$$= \int \frac{y}{2}\left(\frac{dy}{2}\right) + y\,dy + y\,dy$$

$$= \int_{y=2}^{6} \frac{9y}{4}\,dy = \frac{9}{8}\,y^2\Big|_2^6$$

$$\frac{9}{8}\times 36 - \frac{9}{8}\times 4 = 36.$$

or

eq. of line, $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-2}{2} = t$ (say)

$x = t+1$, $y = 2t+1$, $z = 2t$.

$\boxed{dx = dt}$ $\boxed{dy = 2dt}$ $\boxed{dz = 2dt}$

$$I = \int_C (x\,dx + y\,dy + z\,dz) = \int (t+1)\,dt + 2(t+1)\,2dt + 2(t+1)\,2dt$$

$$= 9\left(\frac{(t+1)^2}{2}\right)\Big|_{t=0}^{2}$$

$1 \le x \le 3$
$1 \le t+1 \le 3$
$0 \le t \le 2$

$= 36.$

Ques: find $\int_C \vec{v}\cdot d\vec{r}$, where $\vec{v} = x^2y\,\hat{i} - xy^2\,\hat{j}$
& $\vec{r} = t\,\hat{i} + t^2\,\hat{j}$.
$0 \le t \le 3$.

Sol^n^. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$v \cdot d\vec{r} = x^2 y\, dx - xy^2\, dy.$

Here C is defined as $\vec{r}(t) = t\hat{i} + t^2\hat{j}$

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = t\hat{i} + t^2\hat{j}.$

$\boxed{x = t}$ $\boxed{y = t^2}$, $\boxed{z = 0}$

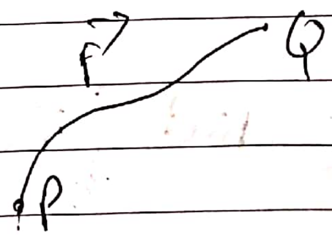
$$I = \int_{t=0}^{3} [t^2 \cdot t^2\, dt - t \cdot t^4 \cdot 2t\, dt].$$

$$= \left(\frac{t^5}{5} - \frac{2t^7}{7}\right)_{t=0}^{3} = \left(\frac{3^5}{5} - \frac{2 \cdot 3^7}{7}\right)$$

* **Application of line Integral:**

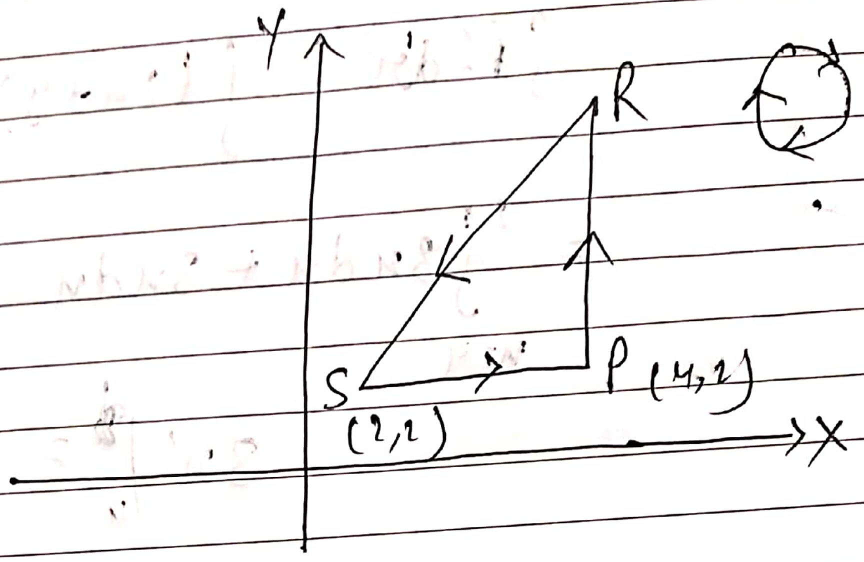
Work done: Let $\vec{F}$ be a force vector then workdone in carrying a particle from P to Q along the curve C is

$$\int_{C:P}^{Q} F \cdot d\vec{r}$$



Ques: Find the work done by force $\vec{F}$ in taking particle from P to Q where $\vec{F}$: $F = (2x+y)\hat{i} + (4y-x)\hat{j}$, C is taken one round the triangle with vertices at (2,2), (4,2), (4,4) in counter clockwise.

Soln



Total work done in taking a particle along the path PRSP : C using force $\vec{F} = (2x+y)\hat{i} + (4y-x)\hat{j}$

$$= \int_{C:PRSP} \vec{F}\cdot d\vec{r} = \int_{PR} \vec{F}\cdot d\vec{r} + \int_{RS} \vec{F}\cdot d\vec{r} + \int_{SP} F\cdot dr \quad \text{--- (1)}$$

$$F\cdot d\vec{r} = (2x+y)dx + (4y-x)dy$$

→ over the PR ⇒ Eq of PR is $x = 4$. ⇒ $dx = 0$

$$\therefore \int_{PR} F\cdot d\vec{r} = \int_{PR} [(2x+y)dx + (4y-x)dy]$$

$$= \int_{y=2}^{4} (4y-4)\,dy = 16.$$

→ over the RS : Eq. of RS is $y = x$.

$$dx = dy$$

$$\int \vec{F}\cdot d\vec{r} = \int (2x+y)dx + (4y-x)dy$$

$$= \int_{x=4}^{2} 3x\,dx + 3x\,dx = \int_{4}^{2} 6x\,dx$$

$$3x^2\Big|_4^2 = 12 - 48$$

$$= \boxed{-36.}$$

over the SP : Eq. of SP is $y=2$ $\boxed{dy=0}$

$$\int \vec{F}\cdot d\vec{r} - \int (2x+y)dx + (4y-x)dy$$

$$= \int_{x=2}^{4} (2x+2)dx = 25 - 9 = 16.$$

Total work done $= \int_{PRSP} \vec{F}\cdot d\vec{r} = 16 - 36 + 16 = -4.$

\# Conservative vector field: A vector field or function $\vec{v}(x,y,z)$ is said to be conservative if $\vec{v} = \text{grad}\,\phi = \vec{\nabla}\phi$ where $\phi(x,y,z)$ is some sclar function.

\# If a vector function $\vec{v}$ is conservative the work done using this vector function is independent of path.

If a vector $\vec{v}$ is Conservative then work done using vector function is zero along a simple closed path.

If $\vec{v}$ is Conservative vector field then
Curl $\vec{v}$ = Curl (grad ϕ) = 0.

Ques. Check given vector field is Conservative or not. If yes, then find potential function.

(i) $\vec{F} = 2xy\hat{i} + (x^2 + 2yz)\hat{j}$

(ii) $\vec{F} = (y^3 - 3x^2 z)\hat{i} + 3xy^2\hat{j} - x^3\hat{k}$.

(iii) $\vec{F} = (2x + ye^{xy})\hat{i} + (2y + xe^{xy})\hat{j}$

Solⁿ. (ii) $\vec{F} = (y^3 - 3x^2 z)\hat{i} + 3xy^2\hat{j} - x^3\hat{k}$.

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 - 3x^2 z & 3xy^2 & -x^3 \end{vmatrix}$$

$$= \hat{i}[0 - 0] - \hat{j}[-3x^2 - (3x^2)] + \hat{k}[3y^2 - 3y^2].$$

$$= 0$$

Curl = 0 then $\vec{F}$ is Conservative. (irrotational)

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Now, let $\vec{F} = \nabla \phi$ = grad ϕ, where ϕ is potential function.

$$(y^3 - 3x^2 z)\hat{i} + 3xy^2 \hat{j} - x^3 \hat{k} =$$

$$\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial x} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y^2 - 3x^2 z \quad \text{— (i)} \qquad \frac{\partial \phi}{\partial y} = 3xy^2 \quad \text{— (ii)}$$

$$\frac{\partial \phi}{\partial z} = -x^3 \quad \text{— (iii)}$$

Integrating (i) w.r.t x.

$$\Rightarrow \phi[x, y, z] = y^3 x - \frac{3x^3 z}{3} + K(y, z) \quad — ☆$$

put in (ii)

$$\text{(ii)} \Rightarrow 3xy^2 - 0 + \frac{\partial K}{\partial y} = 3xy^2 \Rightarrow \frac{\partial K}{\partial y} = 0$$

Integrate w.r.t y.

$$K(y, z) = h(z)$$

put in ☆

$$\phi(x, y, z) = xy^3 + x^3 z + h(z) \quad — ☆☆$$

put in (iii)

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$$0 - x^3 + \frac{\partial h}{\partial z} = -x^3 \Rightarrow \frac{dh}{dt} = 0$$

Integrating w.r.t z

$h(z) = C =$ Constant.

Put in A

$$\boxed{\phi(x,y,z) = xy^3 - x^3 z + C = \text{Potential Form}}$$

Ques: Show that given integral is independent of path and hence evaluate it.

(i) $$\int_{P(-1,2)}^{Q:(2,3)} [2xy^2 dx + (2x^2 y + 1) dy]$$

(ii) $$\int_{P:(1,1,1)}^{Q(-2,-3,-4)} [(3x^2 + 2xyz) dx + (1 + x^2 z) dy + x^2 y dz]$$

(iii) $$\int_{P:(-1,2,3)}^{Q:(2,2,4)} (2xy + z) dx + (x + z) dy + (x^2 + y) dz$$

Sol^n (1).

$$I = \int_P^Q (3x^2 + 2xyz)\,dx + (1+x^2z)\,dy + x^2y\,dz]$$

$$= \int_P^Q [(3x^2 + 2xyz)\hat{i} + (1+x^2z)\hat{j} + x^2y\hat{k}] \cdot [\hat{i}dx + \hat{j}dy + \hat{k}dz]$$

$$= \int_P^Q \vec{F} \cdot dr. \qquad \begin{cases} \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \\ dr = \hat{i}dx + \hat{j}dy + \hat{k}dz \end{cases}$$

where. $\vec{F} = (3x^2 + 2xyz)\hat{i} + (1+x^2z)\hat{j} + x^2y\hat{k}$.

Now, curl; $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2+2xyz & 1+x^2z & x^2y \end{vmatrix}$

$$= \hat{i}[x^2 - x^2] - \hat{j}[2xy - 2xy] + \hat{k}[2xz - 2xz] = 0$$

$\vec{F}$ is Conservative $\Rightarrow$ (1) is independent of path

Since F is Conservative $\Rightarrow \vec{F} = \vec{\nabla}\phi$, where ϕ is sclar potential

$\Rightarrow (3x^2 + 2xyz)\hat{i} + (1 + x^2 z)\hat{j} + x^2 y \hat{k} =$

$\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$.

$\Rightarrow \frac{\partial\phi}{\partial x} = 3x^2 + 2xyz$ — (i)

integrating w.r.t x.

$\phi(x,y,z) = x^3 + x^2yz + k(y,z)$ — ★

$\Rightarrow \frac{\partial\phi}{\partial y} = 1 + x^2 z$ — (ii)

put ★ in (ii)

$\Rightarrow$

$0 + x^2 z + \frac{\partial k}{\partial y} = 1 + x^2 z \Rightarrow \frac{\partial k}{\partial y} = 1$

integrating w.r.t y.

$= k(y,z) = y + h(z)$, put in ★.

(i) ★ $\Rightarrow \phi(x,y,z) = x^3 + x^2yz + y + h(z)$ ★★

put in (iii)

integrating w.r.t z

$h(z) = C =$ Constant

$\cancel{h(z) = C} = C$ put in ★

$\phi(x,y,z) = x^3 + x^2y + y + C$

Now, $I = \int_P^Q \vec{F}\cdot d\vec{r} = \int_P^Q \left(\hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}\right)$

$\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$.

$$= \int_P^Q \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_P^Q d\phi$$

$$\left[\phi(x,y,z)\right]_P^Q = \left[x^3 + x^2 y + y + c\right]_{(1,1,1)}^{(-2,-3,-4)}$$

$$[-8 + 48 - 3 + c] - [3 + c] = 34.$$

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* ~~Green's theorem: Let C be a simple closed curve, bounding a region R. Let~~

Integral $\int_P^Q [f\,dx + g\,dy + h\,dz]$ will be independent of path, if curl $\vec{F} = 0$,

if $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$.

$\Rightarrow$ Curl $\vec{F} = 0 \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x},$

$\cancel{\partial g}\ \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$

* **Green's theorem:** Let C be a simple closed curve bounding a region R, let $M(x,y), N(x,y), \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous over the region R, then

$$\oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Ques: verify green's theorem $\oint (x+y)\,dx + x^2\,dy]$, C is the triangle with vertices (0,0) (2,0) & (2,4) taken in order.

Sol. Green's theorem is $\oint_C (M\,dx + N\,dy) = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, compare $\oint_C [(x+y)\,dx + x^2\,dy]$ with

$$\oint (M\,dx + N\,dy)$$

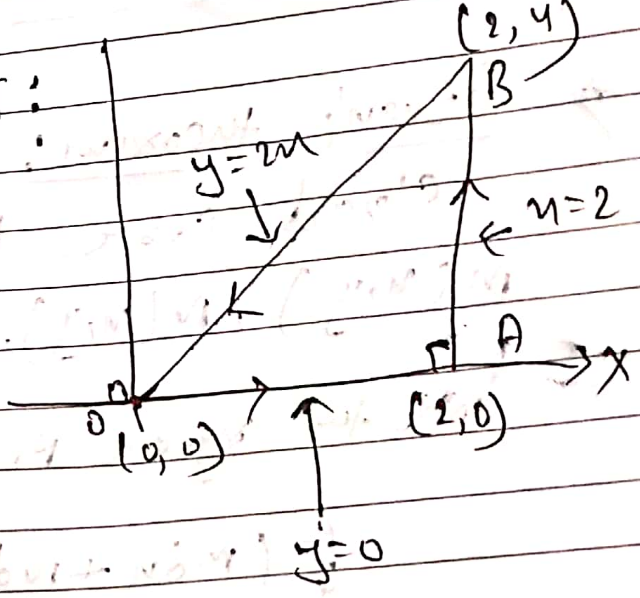
$M = (x+y)$ & $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = 2x$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 1$$

Consider LHS of G.T:

$$\oint_C (M\,dx + N\,dy)$$

C: OABO



$$= \int_{OA} (M\,dx + N\,dy) + \int_{AB} (M\,dx + N\,dy) + \int (M\,dx + N\,dy) \quad \text{—— (1)}$$

over OA: Eq. of OA $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2.

$$\therefore \int_{OA} (M\,dx + N\,dy) = \int_{OA} \left[(x+y)\,dx + x^2\,dy\right]$$

$$= \int_{x=0}^{2} x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{4}{2} = 2$$

over AB eq is $x = 2 \Rightarrow dx = 0$, y varies from 0 to 4.

$$\int_{AB} (M\,dx + N\,dy) = \int_{AB} \left[(x+y)\,dx + x^2\,dy\right]$$

$$= \int_{y=0}^{4} 4\,dy = \underline{16}.$$

over BO: eq is $y = 2x \Rightarrow dy = 2dx$.

$$\int_{BO} (Mdx + Ndy) = \int [(x+y)\,dx + x^2 dy]$$

$$= \int_{x=2}^{0} (x+2x)dx + x^2(2dx)$$

$$= \left(\frac{3x^2}{2} + \frac{2x^3}{3}\right)_{x=2}^{0} = 0 - \left[\frac{3 \times 4}{2} + \frac{2 \times 8}{3}\right]$$

$$= -6 - \frac{16}{3}.$$

Put in (1).

$$\oint_{C: OABO} (Mdx + Ndy) = 2 + 16 - 6 - \frac{16}{3} = 12 - \frac{16}{3} = \frac{20}{3}$$

Now RHS of G-T: $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R (2x-1)\,dx\,dy$$

Limit of R are $0 \le y \le 4$, $\frac{y}{2} \le x \le 2$

$$= \int_{y=0}^{4}\int_{x=y/2}^{2} (2x-1)\,dx\,dy = \int_{y=0}^{4} \left. x^2 - x \right|_{y/2}^{2} dy$$

$$\int_{y=0}^{4} [4-2] - \left[\frac{y^2}{4} - \frac{y}{2}\right] dy = 4 \cdot 8 - \frac{64}{12} + \frac{16}{4}$$

$$= 8 - \frac{4}{3} = \frac{20}{3}$$

Hence LHS = RHS $\Rightarrow$ Green's theorem is verified.

H.W **Ques:** verify G-T for $\oint_C [(xy^2+2xy)\,dx + x^2y\,dy]$, C is the boundary of region enclosing $y^2=4x$; $x=3$.

Ques: Evaluate the line integral using Green theorem.

$$\oint_C (3x^2y\,dx - 2xy^2\,dy), \text{ C is the boundary}$$

of region $x^2+y^2 \le 16$, $x \ge 0$; $y \ge 0$.

soln. G-T is $\oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Compare $\oint_C (3x^2y\,dx - 2xy^2\,dy)$ with $\oint_C (M\,dx + N\,dy)$

$$M = 3x^2y \ \& \ N = -2xy^2$$

$\therefore$ By using G-T $\Rightarrow \oint_C \underbrace{3x^2y\,dx}_{M\,dx} + \underbrace{(-2xy^2)\,dy}_{N\,dy}$

$$= \iint_R \frac{\partial}{\partial x}[-2xy^2] - \frac{\partial}{\partial y}[3x^2y]\,dy$$

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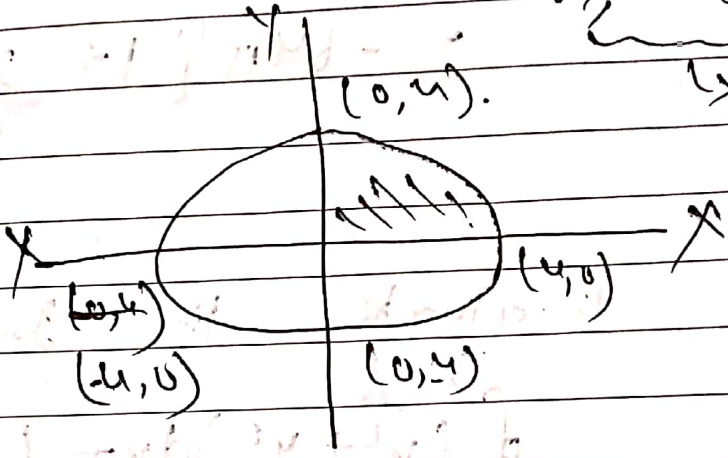
$$= \iint_R \left(-2y^2 - 3x^2\right) dx\, dy$$

$x \geq 0 \quad y \geq 0$ ↳ 1st quad.



Here R is the region bounded by C: OABO and eq. of R is $x^2+y^2 \geq 0,\ y \geq 0$ changing ~~changing~~ to parametric form [Polar Co-ordinate]

$$x = r\cos\theta,\ y = r\sin\theta \Rightarrow dx\,dy = r\,d\theta\,dr$$

Limits of r and θ are $0 \leq r \leq 4,\ 0 \leq \theta \leq \pi/2$

$$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=0}^{4} -\left[2r^2\sin^2\theta + 3r^2\cos^2\theta\right] r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[-2\sin\theta + 3\cos^2\theta\right]\left(\frac{r^4}{4}\right)_{r=0}^{4} d\theta$$

$$= -64\int_{\theta=0}^{\pi/2} 2\left[\frac{1-\cos 2\theta}{2}\right] + 3\left[\frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= -64\left[\left(\theta - \frac{\sin 2\theta}{2}\right)_{\theta=0}^{\pi/2} + \frac{3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_{0}^{\pi/2}\right.$$

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$$-64\left[\left(\frac{\pi}{2}\right)+\frac{3}{2}\left(\frac{\pi}{2}\right)\right]$$

$$= -64\pi\left[1+\frac{3}{2}\right] = -32\pi \times \frac{5}{2}$$

$$= -80\pi.$$

H.W Evaluate line integral using G-T

$$\oint_C \left[(x^2+y^2)dx + (5x^2-3y)dy\right]$$

where C is the boundary of region $x^2=4y$, $y=4$.

Ques: Let C be simple closed curve, enclosing a region R. Use G-T to show that the area of region R

$$R = \oint x\,dy = -\oint y\,dx = \frac{1}{2}\oint (x\,dy - y\,dx)$$

Sol^n: G-T is $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Compare $\oint_C x\,dy$ with $\oint (M\,dx + N\,dy$

$\Rightarrow$ $M=0$, $N=x$

Now, by G.T $\oint x\,dy = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0)\right] dx\,dy$

$= \iint_R dx\,dy$ = Area of region R

Now compare $\oint_c -y\,dx$ with $\oint_c (M\,dx + N\,dy)$

$\Rightarrow M = -y,\ \&\ N = 0$

$\Rightarrow$ By G.T $\oint_c -y\,dx = \iint_R \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-y)\right] dx\,dy$

$= \iint_R dx\,dy$ = Area of region R.

Now, compare $\oint_c \left(\frac{x}{2}dy - \frac{y}{2}dx\right)$ with $\oint (M\,dx + N\,dy)$

$\Rightarrow M = \frac{-y}{2}\ \&\ N = \frac{x}{2}$.

$\therefore$ By G.T $\oint_c \left(\frac{x}{2}dy - \frac{y}{2}dx\right) = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right)\right] dx\,dy$

$= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy = \iint_R dx\,dy$

= Area of region R.

Ques: Use above question & find the area of the circle $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta \le 2\pi$

$x^2 + y^2 = a^2 \rightarrow \pi a^2$.

Sol. As per previous question area of region R which is bounded closed curve C is $\oint_C \frac{1}{2}\{x\,dy - y\,dx\}$

$\therefore$ Area of circle $= \frac{1}{2}\oint_C (x\,dy - y\,dx)$

$$= \frac{1}{2}\oint_C \left[(a\cos\theta)(a\cos\theta\,d\theta) - a\sin\theta(-a\sin\theta\,d\theta)\right]$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} a^2[\cos^2\theta + \sin^2\theta]\,d\theta$$

$$= \frac{a^2}{2} \times 2\pi = \pi a^2.$$

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Surface Integral : Consider $I = \iint_S g(x,y,z)\,dA$

To solve I, we take projection of S on one of the co-ordinates planes.

⇒ **Case : I :** If the projection surface S is taken on x-y plane & let the projection is region R of xy-plane, Then

$$I = \iint_R g(x,y,f(x,y))\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} =$$

where eq of surface S is $z = f(x,y)$
where $\hat{n}$ is unit normal to surface S.

$$\iint g(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}$$

$$f_x \to \left(\frac{\partial f}{\partial x}\right)^2 \qquad f_y \to \left(\frac{\partial f}{\partial y}\right)^2$$

⇒ **Case : 2** Let S be protected onto yz-plane and projection is region R_1 then

$$I = \iint_{R_1} g(k(y,z),y,z)\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \iint_{R_1} g(k(y,z),y,z)\sqrt{1+(k_y)^2+(k_z)^2}$$

$$k_y \to \left(\frac{\partial k}{\partial y}\right)^2 \qquad k_z \to \left(\frac{\partial k}{\partial z}\right)^2$$

Case 3 :

If S is projected on xz plane & let projection is R_2, then

$$I = \iint_{R_2} g(x, h(x,z), z) \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \iint_{R_2} g(x, h(x,z), z) \sqrt{1 + (h_x)^2 + (h_z)^2}$$

$$\left(\frac{dh}{dx}\right)^2 \quad \left(\frac{dh}{dz}\right)^2.$$

where eq. of surface S is $y = h(x,z)$ and $\hat{n}$ is unit normal to surface S.

Area of a surface S is $= \iint_S dA$

Ques: Find the surface of cone $z^2 = x^2 + y^2$, $0 \le z \le 4$

Sol. Surface area of cone $= \iint_S dA$

Taking the projection of surface of cone S onto xy-plane & let this region on xy-plane is R.

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$\therefore$ Area of cone $= \iint_R \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

where $\hat{n}$ is unit normal to surface S:

$$x^2+y^2-z^2=0$$

$$F(x,y,z) = x^2+y^2-z^2$$

$\therefore$ Normal surface $S = \vec{\nabla} F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z} = \vec{N}$

$$\Rightarrow \vec{N} = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-2z)$$

unit normal to surface $S = \frac{\vec{N}}{|\vec{N}|}$

$$= \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2[x\hat{i}+y\hat{j}+z\hat{k}]}{2\sqrt{x^2+y^2+z^2}}$$

$z^2 = x+y^2$

$z = \sqrt{x^2+y^2}$

$$\hat{n}\cdot\hat{k} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+x^2+z^2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{2}\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{2}}$$

$\therefore I = \iint_R \frac{dx\,dy}{(1/\sqrt{2})} = \sqrt{2} \iint_R dx\,dy$

where R is region $x^2+y^2 \le 16$, $z=0$

$= \sqrt{2}\,\pi (4)^2 = 16\sqrt{2}\,\pi$

Flux of vector field $\vec{V}$ through a surface S :

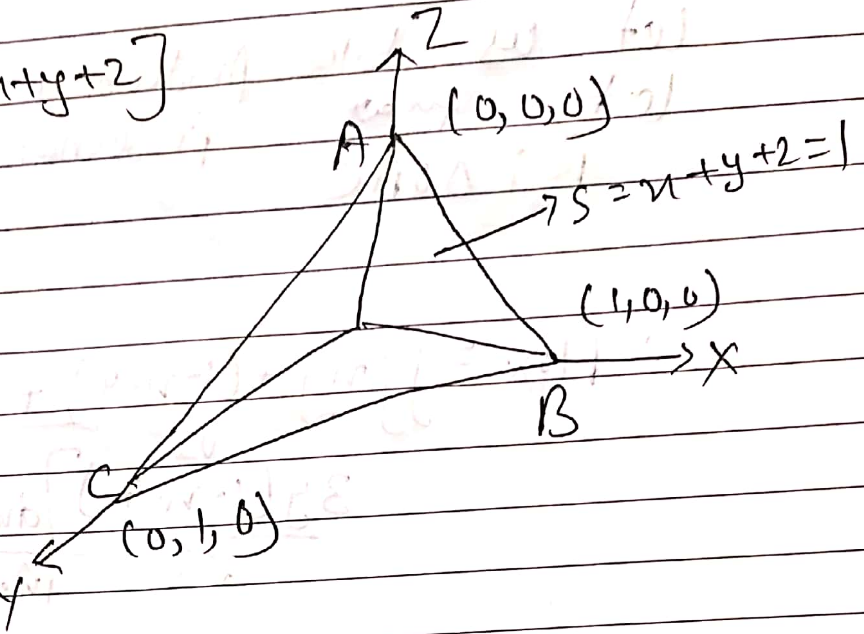
Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be a vector function representing velocity of Fluid, then Flux through a surface S is the volume of fluid flowing through the surface S in unit time.

$$\text{Flux} = \iint \vec{V}\cdot\hat{n}\,dA$$ where $\hat{n}$ is unit normal to surface S.

Ques: Find flux of the vector field $\vec{V} = xy\hat{i} + z\hat{j} + 3xy\hat{k}$ through the surface S, which is portion of plane $x+y+z=1$ included in first octant.

Soln: We know flux by velocity vector $\vec{V}$ through surface S ~~its~~ whose eq is $x+y+z-1=0$ is $\vec{N}$

$\vec{N} = \vec{\nabla}\{x+y+z\}$



$\Rightarrow \vec{N} = \hat{i}\frac{\partial}{\partial x}(x+y+z-1) + \hat{j}\frac{\partial}{\partial y}(x+y+z-1)$

$+ \hat{k}\frac{\partial}{\partial z}(x+y+z-1)$

$= \hat{i} + \hat{j} + \hat{k}$

$\Rightarrow$ Unit normal to surface S is

$= \hat{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1+1+1}} = \frac{\hat{i}+\hat{j}+\hat{k}}{3}$

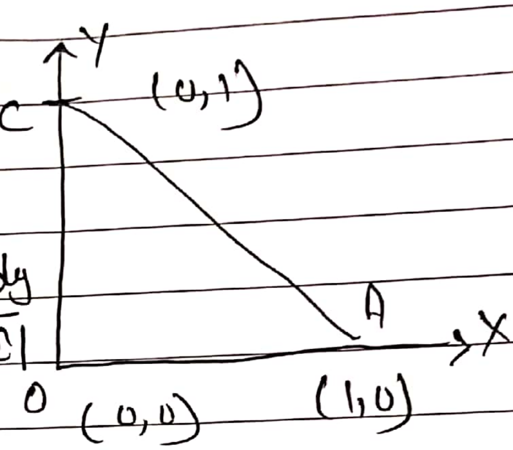
$\vec{v}\cdot\hat{n} = (xy\hat{i} + z\hat{j} + 3yz\hat{k})\cdot\left(\frac{\hat{i}+\hat{j}+\hat{k}}{3}\right)$

$= \frac{xy+z+3yz}{\sqrt{3}}$

$\text{Flux} = \iint_S \vec{v}\cdot\hat{n}\,dA = \iint_S \frac{xy+z+3yz}{\sqrt{3}}\,dA$

Let us take projection of S into XY-plane
Let that projection be termed as region
R : ΔOAC

$$\therefore \text{flux} = \iint \left[\frac{xy + (1-x-y) +}{\sqrt{3}} \frac{3y(1-x-y)}{\sqrt{3}} \right] \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$



$$\hat{n}\cdot\hat{k} = \left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \right) \cdot \hat{k}$$

$$= \frac{1}{\sqrt{3}}$$

$x + y + z = 1$

$z = (1 - x - y)$

$$= \iint_R \frac{xy(1-x-y) + 3y - 3xy - 3y^2)}{\cancel{\sqrt{3}}} \times \frac{dx\,dy}{1/\sqrt{3}}$$

Limits of Region R, $0 \le y \le 1$, $0 \le x \le 1-y$

$$= \int_{y=0}^{1} \int_{x=0}^{1-y} \left[-2xy + 1 - x + 2y - 3y^2 \right] dx\,dy$$

↳ Complete at Your own.

Gauss - divergence theorem :

Let S be a closed surface, occupying a volume V, and let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be a vector function, then

$$\iint \vec{v}\cdot\hat{n}\,dA = \iiint \operatorname{div}\vec{v}\,dx\,dy\,dz \text{ (over } V)$$

where $\hat{n}$ is ~~unit~~ outward unit normal to surface S.

Ques: Let S be a closed surface having volume V

(i) $\iint_S \vec{r}\cdot\hat{n}\,dA = 3V$ & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(ii) $\iint_S \vec{a}\cdot\hat{n}\,dA = 0$, where $\vec{a}$ is constant vector

(iii) $\iint_S \operatorname{curl}\vec{v}\cdot\hat{n}\,dA = 0$

Sol^n Gauss Div. theorem is $\iint_S \vec{v}\cdot\hat{n}\,dA = \iiint$ ~~div~~

$$= \iiint \operatorname{div}\vec{v}\,dx\,dy\,dz$$

(iv) $\iint_S \nabla r^2 \cdot \hat{n}\, dA = 6V$, where $r = |\vec{r}|$

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$= \iint_S \hat{r} \cdot \hat{n}\, dA = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dx\,dy\,dz$

$= 3 \underbrace{\iiint dx\,dy\,dz}_{\text{volume of closed surface } S.}$

$= 3V.$

(ii) By GDT $\iint_S \vec{a} \cdot \hat{n}\, dA = \iiint_V \mathrm{div}(\vec{a})\, dx\,dy\,dz$

$= \iiint_V \left[\frac{\partial}{\partial x}(a_1) + \frac{\partial}{\partial y}(a_2) + \frac{\partial}{\partial z}(a_3)\right] dx\,dy\,dz$

(iii) By GDT $\iint_S \mathrm{curl}\, \vec{v} \cdot \hat{n}\, dA = \iint_V \mathrm{div}(\mathrm{curl}\, \vec{v})\, dx\,dy\,dz$

$= \iiint (0)\, dx\,dy\,dz = 0$

(iv) $\nabla(r^2) = \hat{i}\frac{\partial}{\partial x}(r^2) + \hat{j}\frac{\partial}{\partial y}(r^2) + \hat{k}\frac{\partial}{\partial z}(r^2)$

$= \hat{i}\{2x\} + \hat{j}\{2y\} + \hat{k}\{2z\}$

$= 2\{x\hat{i} + y\hat{j} + z\hat{k}\}$

$= 2\vec{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

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Now $\iint \nabla r^2 \cdot \hat{n}\, dA = \iint_S 2\vec{r} \cdot \hat{n}\, dA$

$= 2\iint_S \vec{r} \cdot \hat{n}\, dA$

$= 2[3V]$ | ∵ by eqn no. (1)

$= 6V$

Ques: If S is a closed surface, having volume V, then show that

$$\iint_S r^n \vec{r} \cdot \hat{n}\, dA = (n+3)\iiint_V r^n\, dV$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

Soln We know Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_V \operatorname{div} \vec{F}\, dV \rightarrow dx\,dy\,dz$$

Consider $\iint_S r^n \vec{r} \cdot \hat{n}\, dA = \iiint_V \operatorname{div}(r^n \vec{r})\, dV.$

[∵ By GDT]

Now, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

Now, $r^n\vec{r} = (x^2+y^2+z^2)^{n/2}\{x\hat{i}+y\hat{j}+z\hat{k}\}$

$$\text{div}[r^n\vec{r}] = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{n/2}x\right]$$

$$+\frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{n/2}y\right] + \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{n/2}z\right]$$

$$= (x^2+y^2+z^2)^{n/2}\cdot 1 + x\cdot\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2x$$

$$+ (x^2+y^2+z^2)^{n/2}\cdot 1 + y\cdot\frac{n}{2}\{x^2+y^2+z^2\}^{n/2-1}\cdot 2y$$

$$+ (x^2+y^2+z^2)^{n/2} + z\cdot\frac{n}{2}\{x^2+y^2+z^2\}^{n/2-1}\cdot 2z$$

$$= 3(x^2+y^2+z^2)^{n/2} + n\{\cancel{x}x^2+y^2+z^2\}^{n/2}$$

$$\text{div}[r^n\vec{r}] = (n+3)(x^2+y^2+z^2)^{n/2}$$

(1) $$\iint_S r^n\vec{r}\cdot\hat{n}\,dA = \iiint \text{div}(r^n\vec{r})\,dv = \cancel{(n+3)\iiint}$$

$$(n+3)\iiint r^n\,dv$$

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Ques: Use Gauss divergence theorem to evaluate to evaluate $\iint_S \vec{v}\cdot\hat{n}\,dA$ where $\vec{v} = 2x^3\hat{i} + 3y^3\hat{j} + z^3\hat{k}$, D is the region bounded by $x^2+y^2+z^2=9$.

Sol. Gauss divergence theorem is $\iint_S \vec{v}\cdot\hat{n}\,dA =$

$$\iiint_V \operatorname{div}\vec{v}\,dx\,dy\,dz$$

$$\therefore \iint_S \vec{v}\cdot\hat{n}\,dA = \iiint_V \operatorname{div}\left[2x^3\hat{i} + 3y^3\hat{j} + z^3\hat{k}\right] dx\,dy\,dz$$

$$= \iiint \left[\frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(3y^3) + \frac{\partial}{\partial z}(z^3)\right] dx\,dy\,dz$$

$$= \iiint_V \left[6x^2 + 9y^2 + 3z^2\right] dx\,dy\,dz$$

surface S is of sphere $x^2+y^2+z^2=9$

Changing to spherical co-ordinates (r, θ, ϕ)

$$x = r\sin\phi\cos\theta,\; y = r\sin\phi\sin\theta,\; z = r\cos\phi$$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$$

Limits of (r, θ, ϕ) are $0 \le r \le 3$,

$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

$$= \int_{\phi=0}^{\pi} \int_{\phi 0}^{2\pi} \int_{r=0}^{3} [6r^2 \sin^2\phi \cos^2\theta + 9r^2 \sin^2\phi \sin^2\theta + 3r^2\cos^2\phi] r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} [6\sin^3\phi \cos^2\theta + 9\sin^3\phi \sin^2\theta + 3\cos^2\phi \sin\phi] \left(\frac{r^5}{5}\right)_{r=0}^{3} d\theta\, d\phi$$

$$= \frac{3^5}{5} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left[6\sin^3\phi \left[1 + \frac{\cos 2\theta}{2}\right] + 9\sin^3\phi \left[\theta - \frac{\sin^2\theta}{2}\right] + 3\cos^2\phi \sin\phi \cdot \theta\right]_{\theta=0}^{2\pi} d\phi$$

$$= \frac{3^5}{5} \int_{\phi=0}^{\pi} 3\sin^3\phi [2\pi] + \frac{9}{2} \sin^3\phi [2\pi] + 3\cos^2\phi \sin\phi (2\pi)] d\phi.$$

$$= \frac{3^5}{5} (2\pi) \int_{\phi=0}^{\pi} \frac{15}{2} \sin^3\phi - 3(\cos\phi)^2 (-\sin\phi)] d\phi$$

$$= \frac{3^5}{5} (2\pi) \left[\frac{15}{8}\left[-3\cos\phi + \frac{\cos 3\phi}{3} - \frac{3\cos^3\phi}{3}\right]_{\phi=0}^{\pi}\right.$$

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$$= \frac{3^5}{5}[2\pi]\left[\frac{15}{8}\left[-3(-1)+\frac{1}{3}(-1)-(-1)\right]\right]$$

$$\frac{-15}{8}\left[-3\times 1 - 1/3 - 1\right]$$

$$= \frac{3^5}{5} \times 2\pi \times \frac{15}{8}\left[3-\frac{1}{3}+1+3-\frac{1}{3}+1\right]$$

$$= \frac{3^6\pi}{4}\left[8-\frac{2}{3}\right] = \frac{3^6\pi}{4}\times\frac{22}{3} = \frac{3^5\pi\times 11}{2}$$

Ques: Evaluable surface integral using gauss divergence theorem

$$\iint_S \left[yz\,dy\,dz + 2x\,dz\,dx + xy\,dx\,dy\right]$$

where S is Cube. $0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1.$

Sol. Gauss divergence theorem is

$$\iint_S \vec{v}\cdot\hat{n}\,dA = \iiint_V \operatorname{div}\vec{v}\,dx\,dy\,dz$$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then Gauss divergence theorem is written as

$$\iint_S \left[v_1\,dy\,dz + v_2\,dx\,dz + v_3\,dx\,dy\right]$$

$$= \iiint_V \operatorname{div} \vec{v}\, dx\,dy\,dz$$

Now, $\iint_S [yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy]$

$$= \iiint_V \operatorname{div} [yz\hat{i} + zx\hat{j} + xy\hat{k}]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy)\right] dx\,dy\,dz$$

$$= \iiint_V 0 + 0 + 0\, dx\,dy\,dz = 0.$$

Ques: Evaluate $\iint_S [xy\,dy\,dz + yz\,dz\,dx + zx\,dx\,dy]$ by using gauss - divergence theorem. where C is surface of parallelopiped $0 \le x \le 4,\ 0 \le y \le 3,\ 0 \le z \le 4$

Sol: G.D.T is $\iint_S \vec{v}\cdot\hat{n}\, dA$

$$= \iiint_V \operatorname{div} \vec{v}\, dx\,dy\,dz$$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$= \iint_S [v_1\,dy\,dz + v_2\,dz\,dx + v_3\,dx\,dy] = \iiint_V \operatorname{div} \vec{v}\, dx\,dy\,dz$$

$$\iint_S \left[xy\,dy\,dz + yz\,dz\,dx + zx\,dx\,dy\right] =$$

$$\iiint \mathrm{div}\left[xy\hat{i} + yz\hat{j} + zx\hat{k}\right] dx\,dy\,dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)\right] dx\,dy\,dz$$

$$= \int_{z=0}^{4}\int_{y=0}^{3}\int^{4} \left[y + z + x\right] dx\,dy\,dz$$

$$= \int_{z=0}^{4}\int_{x=0}^{3} \left[4y + 4z + 8\right] dy\,dz$$

$$= \int_{z=0}^{4} \left[\frac{4y^2}{2} + 4zy + 8y\right]_{y_0}^{3} dz$$

$$= \int_{z=0}^{4} \left[18 + 12z + 24\right] dz$$

$$= 168 + 96 = 264$$

→ Stokes theorem gives a relation b/w line integral & surface integral.

→ Stokes theorem is generalization of Green's theorem in three dimension.

* **Stokes theorem:** Let S be the open surface, bounded by a simple closed curve C, let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector function then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dA$$

where $\hat{n}$ is unit normal to surface S.

→ Green's thm is special / particular case of stokes thm in two dimension.

Ques: By using stokes thm prove that $\oint \vec{r}\cdot d\vec{r} = 0$

A. Stokes theorem is $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dA$

$\therefore$ By stokes thm. $\oint_C \vec{r}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{r}\cdot\hat{n}\,dA$

$$= \iint_S \vec{0}\cdot\hat{n}\,dA$$

$$= 0.$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\text{curl}\,\vec{r} = \vec{\nabla}\times\vec{r}$

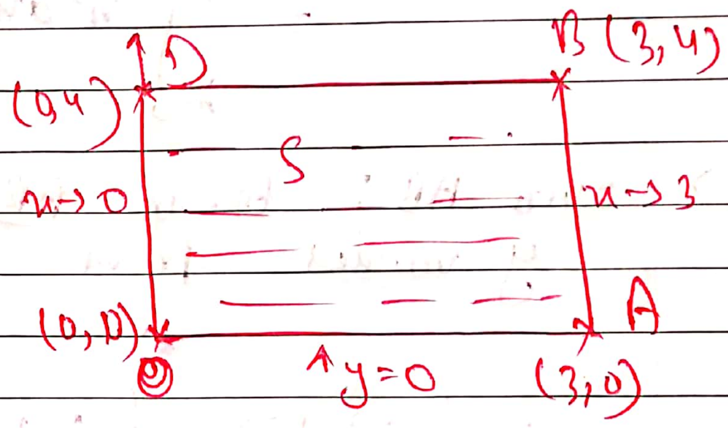
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$\hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0]$

0

Ques: Verify Stokes theorem for $\vec{v} = x^2\hat{i} + x^2y\hat{j}$ & C is the boundary of rectangle where sides are $x=0$, $x=3$, $y=0$, $y=4$, in $z=0$ plane.

Sol. Stokes theorem is $\oint \vec{v}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{v}\cdot\hat{n}\, dA$



Now, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$\vec{v}\cdot d\vec{r} = (x^2\hat{i} + x^2y\hat{j})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$= x^2dx + x^2y\,dy$.

Now, L.H.S of S.T : $\oint_{C:\,OABDO} \vec{v}\cdot d\vec{r}$

$= \int_{OA} \vec{v}\cdot d\vec{r} + \int_{AB} \vec{v}\cdot d\vec{r} + \int_{BD} \vec{v}\cdot d\vec{r} + \int_{DO} \vec{v}\cdot d\vec{r}$

→ over OA : Eq. of OA; $y = 0$, $\Rightarrow dy = 0$

x varies from 0 to 3

$$\int_{OA} \vec{v}\cdot d\vec{r} = \int_{OA} \left[x^2\,dx + x^2 y\,dy\right]$$

$$= \int_{x=0}^{3} x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9.$$

→ over AB : Eq. of AB is $x = 3 \Rightarrow dx = 0$,
y varies from 0 to 4.

$$\therefore \int_{AB} \vec{v}\cdot d\vec{r} = \int_{AB} \left(x^2\,dx + x^2 y\,dy\right)$$

$$= \cancel{\int_{x=0}^{3} x^2\,dx} = \int_{y=0}^{4} 9y\,dy = \frac{9y^2}{2}\Big|_0^4 = 72$$

over BD : Eq. of BD $y = 4 \Rightarrow dy = 0$
x varies from 3 to 0.

$$\therefore \int_{BD} \vec{v}\cdot d\vec{r} = \int_{BD} \left[x^2\,dx + x^2 y\,dy\right] = 0$$

$$= \int_{x=3}^{0} x^2\,dx = \frac{x^3}{3}\Big|_3^0 = 0 - 9 = -9.$$

Over DO : Eq. is $x=0 \Rightarrow dx=0$, y varies from 4 to 0.

$$\therefore \oint_{DO} \vec{v}\cdot d\vec{r} = \int_{DO} [x^2 dx + x^2 y\, dy] = \int_{D} 0 = 0$$

$$\int_{C:\, OABDO} \vec{v}\cdot d\vec{r} = 9 + 72 - 9 + 0 = 72$$

RHS of Stokes thm: $\iint_S \text{curl}\, \vec{v}\cdot \hat{n}\, dA$

Consider curl $\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & x^2 y & 0 \end{vmatrix}$

$$= \hat{i}[0-0] + (-\hat{j})[0-0] + \hat{k}[2xy - 0]$$

$$= 2xy\hat{k}.$$

Here, surface S is part of XY-plane so unit normal to S is $\hat{k} = \hat{n}$

$\therefore$ ~~curl~~ $\text{curl}\,\vec{v}\cdot\hat{n} = 2xy\,\hat{k}\cdot\hat{k} = 2xy$

$$\therefore \iint_S (\text{curl}\,\vec{v}\cdot\hat{n})\, dA = \int_{y=0}^{4}\int_{x=0}^{3} 2xy\, dx\, dy$$

$y=0 \qquad x=0$

$$= 2\left[\frac{x^2}{2}\right]_{x=0}^{3}\left[\frac{y^2}{2}\right]_{y=0}^{4}$$

$$= 2\left[\frac{9}{2}\right]\left[\frac{16}{2}\right] = 72.$$

Hence $\oint_{C:\,OABDO} \vec{v}\cdot dl = \iint_S \text{curl}\,\vec{v}\cdot\hat{n}\,dA$,

Hence Stokes theorem get verified.

Ques: Using Stokes theorem $\oint \vec{v}\cdot d\vec{r}$, where $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ C is the boundry of ellipsoid $y = \frac{\sqrt{144-36x^2-9z^2}}{4}$ in $y=0$ plane.

Sol. We know Stokes ~~theorem~~ thm. is

$$\oint \vec{v}\cdot dr = \iint_S \text{curl}\,\vec{v}\cdot\hat{n}\,dA$$

where $\hat{n}$ is unit normal to surface S.

$\therefore$ By Stokes thm $\oint_C \vec{v}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{v}\cdot\hat{n}\,dA$

Now, curl $\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$\hat{i}[0-0] + \hat{j}[0-0] + \hat{k}[0-0]$

$= 0.$

$\therefore \oint \vec{v} \cdot d\vec{r} = \iint_S \text{curl } \vec{v} \cdot \hat{n}\, dA = \iint_S \vec{0} \cdot \hat{n}\, dA = 0$

Line Integral with respect to arc length parameter.

$I = \int_C f(x, y, z)\, ds$ ↳ arc length of parameter

$= \int_C f(x,y,z) \frac{ds}{dt} = \int_C f(x,y,z) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

where eq. of the curve is given parameter form & the parameter is t.

Ques: let $f(x,y,z) = 2x + 3y$, C is given as $x = t$, $y = 2t$, $z = 3t$, $3 \le t \le 0$.

find $\int_C f(x,y,z)\, ds$

Sol. $I = \int_C F(x,y,z)\,ds = \int_C F(x,y,z)\frac{ds}{dt}$

$= \int_C F(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

$= \int_C F(x,y,z)\sqrt{1+4+9}\,dt$

$= \sqrt{14}\int_C (2x+3y)\,dt$

$= \sqrt{14}\int_{t=0}^{3}(2t+6t)\,dt = \sqrt{14}\,8\frac{t^2}{2}\Big|_{t=0}^{3}$

$= 36\sqrt{14}$

Ques. Evaluate $\int_C F(x,y,z)\,dz$ where $R(x,y,z) = x^2y^2z$, C is line segment joining $(1,2,2)$ to $(2,3,5)$

Sol. C is line segment joining $(1,2,2)$ to $(2,3,5)$, so its eq is

$\frac{x-1}{2-1} = \frac{y-2}{3-2} = \frac{z-2}{5-2} \Rightarrow \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-2}{3} =$ (say) t

$x = 1+t,\; y = t+2,\; z = 3t+2$

$$I = \int_C F(x,y,z)\, ds = \int_C xy^2 z \frac{ds}{dt}\, dt$$

$$= \int xy^2 z \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$I = \int_C xy^2 z \sqrt{1+1+9}\; dt = \sqrt{11} \int_C xy^2 z\, dt$$

$$= \int_{t=0} (1+t)(t+2)^2 (3t+2)\, dt$$

{Complete at your own}